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# Setting the scale for predictions of asymptotic freedom*

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Asymptotically free theories make explicit predictions for the $q^2$ dependence of the structure-function moments at present energies. The onset of scaling and the deviation from naive scaling determine the only two free parameters for a particular choice of gauge group and fermion representations.

The approximate scaling predictions of asymptotically free theories of the hadrons have been presented as asymptotic functional forms for $q^2 \to \infty$, where the expansion parameter is essentially $[\ln(-q^2/M^2)]^{-1}$, independent of $g$, the initial coupling constant.[1-3] But these theories make unequivocal predictions for SLAC and NAL energies. We have neither the need nor the luxury of taking infinite-momentum limits, and the scheme will either stand or fall in confrontation with accessible data.[4]

The renormalization-group equation[5] states that $g$ and $M$, the normalization scale, are not independent variables in the sense that a renormalized field theory depends only on the single combination $\bar{g}=\bar{g}(g, t)$, where $t=-\ln M$. If scaling is observed because $\bar{g}$ is approaching the fixed point at the origin as $t$ increases, the appropriate expansion parameter is $\bar{g}^2/4\pi$. [The rapid onset of scaling is still a strong-coupling problem—the behavior of $\beta(g)$ for strong $g$.] In the scaling region $\bar{g}^2/4\pi$ is small and should be well approximated by the perturbative expression $\bar{g}^2/4\pi=g^2/[4\pi(1+2bg^2t)]$. For the study of momentum dependence in the approximate scaling region, the choice of $t$ or $M$ is now pure convention. $M$ can be any value in the scaling domain, and different $M$'s would correspond to different $g$'s. As good a choice as any is to let $t=0$ mark the onset of scaling. For deep-inelastic scattering $t=\frac{1}{2}\ln(-q^2/M^2)$ and $M^2=4m_{\text{proton}}^2$, or thereabouts. $g$ is now a measure of the departure from naive scaling and must be fitted from the data. The remaining parameters are all group-theoretic.

It is trivial to put the previous predictions[1-3] in a useful form. The momentum dependence of the structure-function moments comes from the anomalous dimension, $\gamma_n$, of the appropriate Wilson coefficient function. If $\gamma_n(\bar{g})=-d_n\bar{g}^2+\cdots$, then

$$\exp\left[\int_0^t \gamma_n(\bar{g})\,dt'\right] = (\bar{g}^2/g^2)^{+d_n/2b} = (1+2bg^2t)^{-d_n/2b}.$$

So the $[\ln(-q^2/M^2)]^{-d_n/2b}$ of the $q^2\to\infty$ analysis should be replaced by $[1+bg^2\ln(-q^2/M^2)]^{-d_n/2b}$.

[A prediction for the $e^+e^-$ total cross section[3] can be gotten from the hadronic contribution to the photon inverse propagator, $\Delta^{-1}(p^2)$. To lowest order in $\bar{g}$

$$\Delta^{-1}=p^2\exp\left[-2a\sum Q^2\int_0^t \bar{e}^2(1+B\bar{g}^2)\,dt'\right],$$

where $e$ is the electromagnetic coupling. For any experimentally accessible $t$, $\bar{e}^2\approx e^2$, which is to say that standard perturbation theory is still good for quantum electrodynamics (QED). Now, expand to lowest order in $e^2$, continue to timelike $p^2$, take the imaginary part, and normalize to $\mu^+\mu^-$. Then

$$\frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} = \sum Q^2[1+B\bar{g}^2+O(\bar{g}^4)] = \sum Q^2\left[1+\frac{Bg^2}{1+bg^2\ln(p^2/M^2)}+\cdots\right],$$

with $B=3c_3/16\pi^2$. But the continuation from spacelike $p^2<0$ to $p^2>0$ makes the whole prediction suspect because small deviations from scaling for $p^2\ll 0$ can lead to very large deviations when $p^2\gg 0$. To see this, consider the dispersion relation for the hadronic vacuum polarization

$$\Pi(p^2)=\Pi(0)+\frac{p^2}{\pi}\int_0^\infty \frac{dz\,\mathrm{Im}\Pi(z)}{z(z-p^2)}.$$

Evidently, $\Pi(p^2\ll 0)$ is rather insensitive to what $\mathrm{Im}\Pi(z\gg 0)$ is doing. So while we may believe the predictions in the deep Euclidean region, $\sigma_{e^+e^-\to\text{hadrons}}$, which is proportional to $\mathrm{Im}\Pi(z>0)$. is not really calculable. In more conventional perturbative problems, e.g., electrodynamics, the straightforward continuation is made and is valid because the particle and threshold structure is taken as known. But there is no such information available for symmetric non-Abelian gauge theories.]

For the structure-function moments, each function of $q^2$ is multiplied by an unpredicted constant: the hadron matrix element of the appropriate composite local parton operator. In the simplest of models, an SU(3) gauge group with three triplets of fermions, this complicates the analysis of the proton moments, or any other moments that have physical-SU(3) singlet components, because there are two different functions of $q^2$ which occur in the singlet component with an unknown ratio, due to the mixing of gluon and fermion operators. For physical-SU(3) nonsinglet moments, e.g., the proton-neutron difference, only a single function occurs.[2]

The first moments appear[6] to be virtually $q^2$-independent and therefore compatible with $g=0$, but the predicted dependence is very weak. The higher moments, tabulated through the eighth, each decrease faster with $q^2$. In a given model, the observed variation sets upper and lower limits on $g$. For a typical "colored" quark-gluon model, $g^2/4\pi$ is on the order of $\frac{1}{10}$.

A serious experimental fit to the moments should be attempted, with the following kept in mind: Higher moments may have pronounced $q^2$ dependence but have inherently greater experimental uncertainty. Physical-SU(3) nonsinglet terms are predicted to have a single momentum dependence. Also, the first moment of physical-SU(3) singlet functions has two components, but one of these is a constant whose value is, in fact, predicted.[7]

## APPENDIX

If a dimensionless function $f$ satisfies a renormalization-group equation

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+\gamma(g)\right]f(q,g,M)=0,$$

it has the form

$$f(\lambda q_0,g,M)=f(q_0,\bar{g},M)\exp\left[\int_0^t\gamma(\bar{g})\,dt'\right],$$

where $t=\ln\lambda$, $\partial\bar{g}/\partial t=\beta(\bar{g})$, and $\bar{g}(g,t=0)=g$. For a simple non-Abelian gauge theory coupled to fermions

$$\beta(g)=-bg^3+\cdots=-\frac{1}{16\pi^2}\left(\tfrac{11}{3}c_1-\tfrac{4}{3}c_2\right)g^3+\cdots,$$

where $c_1\delta_{ab}=f_{acd}f_{bcd}$, $c_2\delta_{ab}=\mathrm{tr}(T^aT^b)$, and $c_3\underline{1}=T^aT^a$, with $f_{abc}$ the structure constants and $T^a$ the fermion representation matrices. For three fermion triplets in SU(3), $c_1=3$, $c_2=\frac{3}{2}$, $c_3=\frac{4}{3}$; for four triplets, $c_2$ becomes 2.

The structure-function moments are

$$\int_0^1 dx\,x^{n-2}F(q^2,x)\approx\sum_i A_n^i[1+bg^2\ln(-q^2/M^2)]^{-d_n^i/2b}$$

for even $n$, where $F$ is $xW_1$ or $\nu W_2$. The constants $A_n^i$ must be fitted. If physical SU(3) is taken to be a global symmetry between fermion multiplets, the physical-SU(3) nonsinglet moments are given by a single

$$\frac{d_n}{2b}=\frac{3c_3}{11c_1-4c_2}\left[1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\right].$$

(Note that for a fixed interval in $q^2$ the total variation of successive moments are related in a model-independent way.) $i$ takes on two values for the singlet functions, and $d_n^i/2b$ are the eigenvalues of the matrix

$$\frac{3}{11c_1-4c_2}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

where

$$\alpha=c_1\left[\frac{1}{3}-\frac{4}{n(n-1)}-\frac{4}{(n+1)(n+2)}+4\sum_{j=2}^{n}\frac{1}{j}\right]+\tfrac{4}{3}c_2,$$

$$\beta=c_3\left[\frac{1}{n+1}+\frac{2}{n(n-1)}\right],$$

$$\gamma=c_2\left[\frac{8}{n+2}+\frac{16}{n(n+1)(n+2)}\right],$$

$$\delta=c_3\left[1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\right].$$

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[1]H. Georgi and H. D. Politzer, Phys. Rev. D 9, 416 (1974).

[2]D. J. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974).

[3]T. Appelquist and H. Georgi, Phys. Rev. D 8, 4000 (1973); A. Zee, Phys. Rev. D 8, 4038 (1973).

[4]Irrespective of its relevance to scaling phenomena, asymptotic freedom holds a unique position for its short-distance calculability in perturbation theory independent of the coupling constant.

[5]The notation of Ref. 1 is used. An appendix summarizes relevant results.

[6]E. D. Bloom, in Proceedings of the Bonn Conference, 1973 (unpublished).

[7]It is the free-quark prediction reduced by the factor $c_3/(2c_2+c_3)$.